CARINGBAH HIGH SCHOOL

2024 HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION

Mathematics Advanced

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen.
- Calculators approved by NESA may be used.
- Reference sheet is provided.
- For questions in Section II, show relevant mathematical reasoning and/or calculations.

Section I – 10 marks (pages 3 - 7)

- Attempt Questions 1–10
- Allow about 15 minutes for this section.
- Answer the questions on page 40.

Section II – 90 marks (pages 10 – 34)

- Attempt Questions 11–32
- Allow about 2 hours and 45 minutes for this section.

Student Number: ______________________Class: ________________

Marker's Use Only								
Section I	**Section II**							**Total**
Q1-10	**11 - 15**	**16 - 18**	**19 - 22**	**23 - 25**	**26- 28**	**29 - 30**	**31-32**	
/10	/12	/13	/16	/13	/11	/10	/15	/100

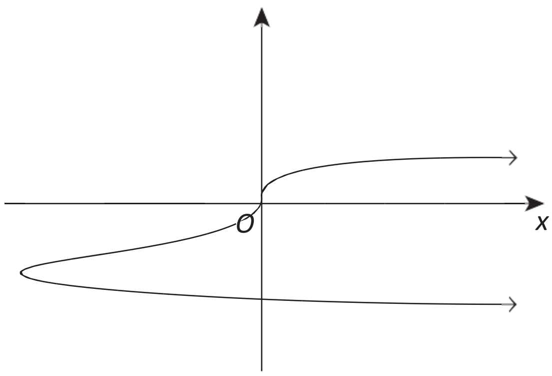
Section I

10 marks
Attempt Question 1-10
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1-10.

---

1 The graph of a relation is shown.



Which type of relationship does the graph represent?

(A) one-to-one

(B) one-to-many

(C) many-to-one

(D) many-to-many

2 Consider the graph $y = 4x + 1$

What is the equation of the graph after it has been translated 2 units to the right?

(A) $y = 2x - 7$

(B) $y = 2x + 9$

(C) $y = 4x - 7$

(D) $y = 4x + 9$

3 Which of the following is the domain of the function $y = \dfrac{1}{\sqrt{x^2 - 1}}$?

(A) $(-\infty,-1]\cup[1,\infty)$

(B) $(-\infty,-1)\cup(1,\infty)$

(C) $(-1,1)$

(D) $[-1,1]$

4 Families in Caringbah were surveyed about the types of pets they own. The results of the survey are shown below.

- 40% of families own a dog.
- 36% of families own a cat.
- 18% of families own both a cat and a dog.

A family is chosen at random and found to own a cat.

What is the probability that the family also owns a dog?

(A) 18%

(B) 36%

(C) 40%

(D) 50%

5 A supermarket assistant sets up a display of lemonade cans. The display forms a triangle, where each row has one fewer can than the one below. There are 15 cans at the base of the display and one can at the top of the display.

What is the total number of cans used in the display?

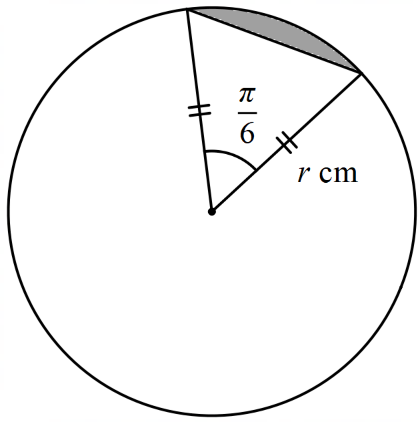
(A) 30

(B) 120

(C) 160

(D) 180

6 The diagram shows a circle with radius r cm and a sector with an internal angle of $\frac{\pi}{6}$ radians.



NOT TO SCALE

If the area of the shaded segment is 1.7 cm^2, which of the following is closest to the radius of the circle?

(A) 0.12 cm

(B) 0.34 cm

(C) 12.00 cm

(D) 144.08 cm

7 Alicia and Steven are participating in a fitness program. The probability that Alicia will complete the program is 0.8, and the probability that Steven will complete the program is 0.7.

What is the probability that only one of them will successfully complete the program?

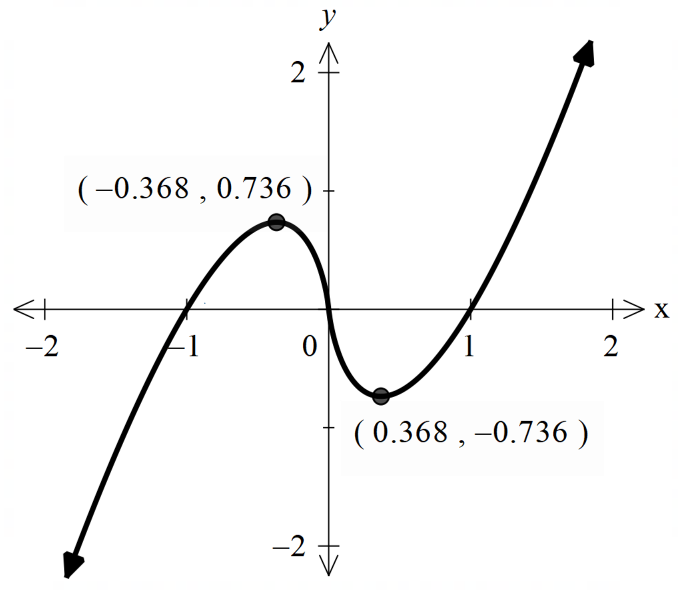
(A) 0.14

(B) 0.24

(C) 0.38

(D) 0.56

8 The graph of $y = x\ln(x^2)$ is shown.



How many solutions does the equation $|x\ln(x^2)| = \frac{1}{3}(x+1)$ have?

(A) 3

(B) 4

(C) 5

(D) 6

9 The following cumulative frequency table shows heights of students at a school.

Height (cm)	Cumulative Frequency
131 – 140	70
141 – 150	210
151 – 160	420
161 – 170	680
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What is the height of a student in the 7^{th} decile?

(A) 131 – 140 cm

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(C) 161 – 170 cm

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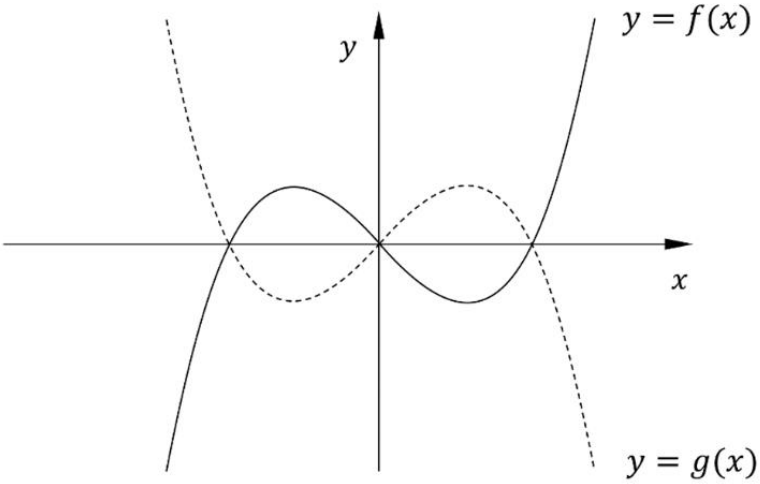
10 The graph shows two cubic functions, $y = f(x)$ and $y = g(x)$.



It is given that $f(x)$ is an odd function and that $g(x) = f(-x)$.

Define $h(x) = f(x) - g(x)$.

How many stationary points does $y = h(x)$ have?

(A) 0

(B) 1

(C) 2

(D) 3

End of Section I

Marks

Question 11

Differentiate $y = xe^{7x}$, expressing your answer in factorised form. **2**

Question 12

Find the exact gradient of the tangent to the curve $y = e^{2x+3}$ at the point where $x = 0$. **2**

Question 13

Solve the equation $\frac{\text{cosec}x}{5} - 2 = 0$ for $0° \leq x \leq 360°$, correct to the nearest minute. **3**

Question 14

A bag contains six cards that are each printed with one of the following numbers.

111, 121, 122, 211, 221, 222

A student chooses one card at random.

Let N_1 be the event that the first digit of the number on the card is 1, and N_2 be the event that the second digit of the number on the card is 1.

Determine whether N_1 and N_2 are independent events. **3**

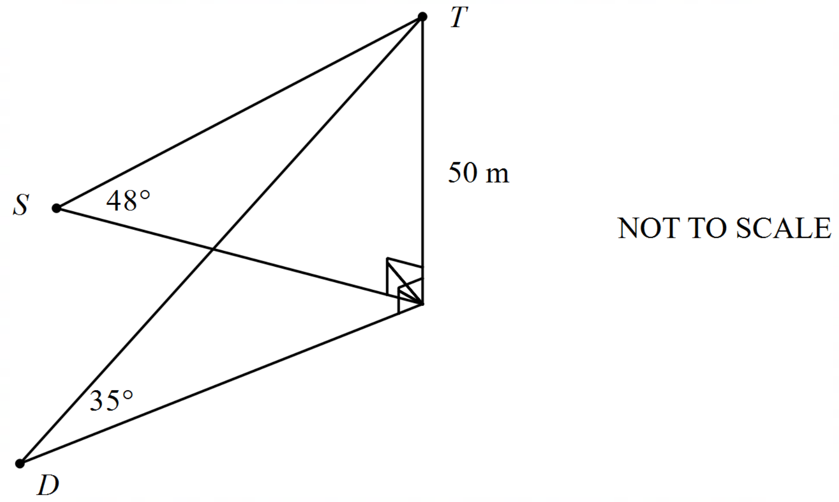
Question 15

Evaluate $\int_0^{\frac{\pi}{4}} \cos(2x)\, dx$ **2**

Question 16

Darren and Sylvia visit Sydney Harbour on New Year's Eve to watch the fireworks.

Darren is standing outside the Sydney Opera House and Sylvia is on a yacht in the harbour. They can both see the fireworks at the top of the Sydney Harbour Bridge.

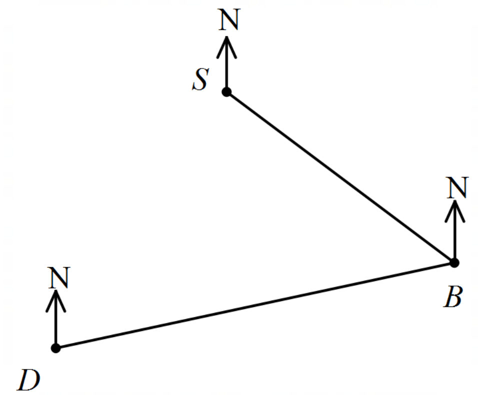


From Darren's position (D), the Sydney Harbour Bridge (B) is at a bearing of $78°$, and the angle of elevation to the top of the bridge (T) is $35°$.

From Sylvia's position (S), the Sydney Harbour Bridge is at a bearing of $140°$, and the angle of elevation to the top of the bridge is $48°$.

The top of the bridge is 50 m above the base of the bridge.

a) Using appropriate diagrams and reasoning, show that $\angle SBD = 62°$. **2**



Question 16 continues on page 13

b) Hence, find the distance between Sylvia and Darren, correct to 2 decimal places. **3**

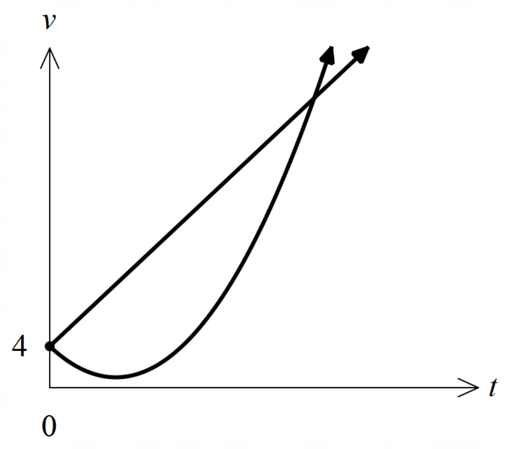
End of Question 16

Question 17

Find the equation of the normal to the curve $y = \dfrac{\ln(x-1)}{x-1}$ at the point $(2,0)$. **3**

Question 18

The velocity of two racing cars, P and Q, are shown on the graph below.



NOT TO SCALE

Question 18 continues on page 15

The velocity of car P at time t seconds is given by the function $v_P = 3t^2 - 6t + 4$, where v_P is the velocity in metres per second.

The velocity of car Q is accelerating at a constant rate. Both cars start at the same point and have the same velocity at times where $t = 0$ and $t = 4$.

a) Show that the equation for the velocity of car Q, v_Q, is given by $v_Q = 6t + 4$. **2**

b) Both cars start the race from the same point. **3**

Find the earliest time when car P will pass car Q after the race starts.

End of Question 18

Question 19

a) Find the derivative of $y = \tan(x^2)$. **1**

---

b) Hence, or otherwise, find $\int x\tan^2(x^2)\,dx$. **3**

Question 20

Use two applications of the trapezoidal rule to find an approximate value of $\int_1^2 \ln x\,dx$. **2**

Give your answer correct to 2 decimal places.

Question 21

a) Show that $\dfrac{1}{\sec\theta - \tan\theta} - \dfrac{1}{\sec\theta + \tan\theta} = 2\tan\theta$ **2**

---

b) Hence or otherwise, find $\displaystyle\int \frac{1}{\sec\theta - \tan\theta} - \frac{1}{\sec\theta + \tan\theta}\, d\theta$ **2**

Question 22

A continuous random variable has the probability density function $f(x)$ given by:

$$f(x)=\begin{cases} \dfrac{kx}{5-x^2} & 0 \le x \le 2 \\ 0 & \text{for all other values of } x \end{cases}$$

a) Show that the value of k is $\dfrac{2}{\ln 5}$. **2**

Question 22 continues on page 19

b) Show that the cumulative distribution function $F(x)$ is given by: **2**

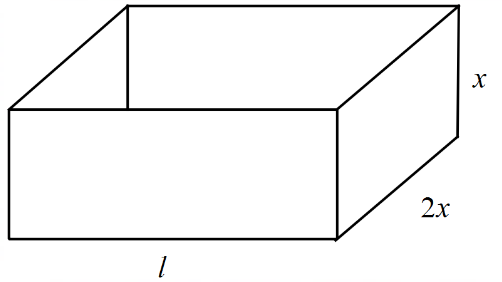
$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - \log_5(5 - x^2) & 0 \le x \le 2 \\ 1 & x > 2 \end{cases}$$

Question 22 continues on page 20

c) Hence, or otherwise, show that the median is $\sqrt{5-\sqrt{5}}$. **2**

Question 23

A factory produces boxes in the shape of an open rectangular prism (without a lid).



NOT TO SCALE

The length, height and width of the box are l cm, x cm and $2x$ cm respectively.

The factory uses k cm^2 of sheet metal to make the box, where k is a constant.

a) Show that $l = \dfrac{k-4x^2}{4x}$. **1**

Question 23 continues on page 21

b) Hence, find the maximum volume of the box that can be produced from a sheet of metal with an area of 1200 cm^2. **3**

End of Question 23

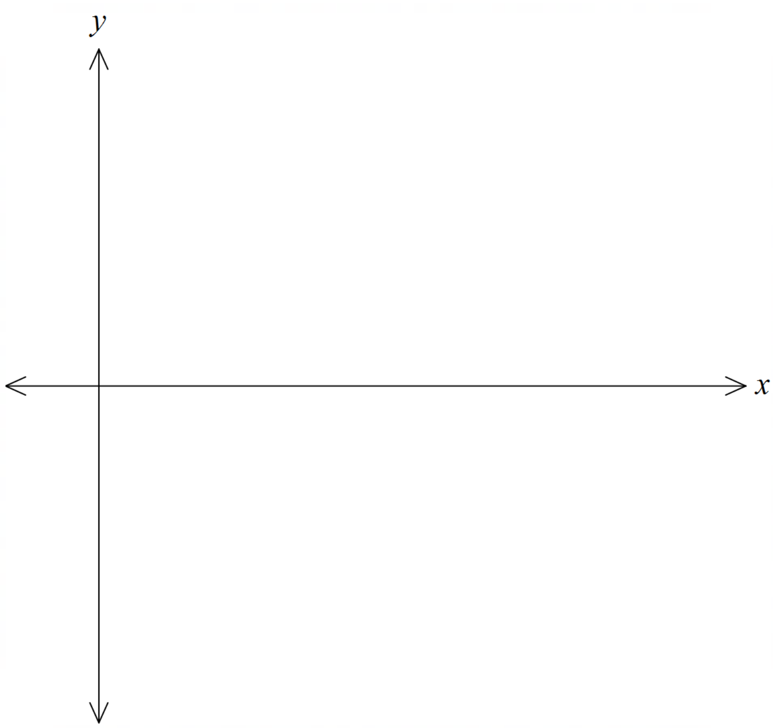
APPROXIMATELY HALFWAY – 55 marks out of 100 complete at this point.

Question 24

Consider the series $\ln x - 3(\ln x)^2 + 9(\ln x)^3 - 27(\ln x)^4 + \ldots$

a) Show that this series is a geometric series. **2**

b) On the set of axes below, sketch the graphs of $y = \ln x$, $y = -\frac{1}{3}$ and $y = \frac{1}{3}$, showing all points of intersection and intercepts with the coordinate axes. **2**



Question 24 continues on page 23

c) Hence, or otherwise, find the values of x for which the series has a limiting sum. **2**

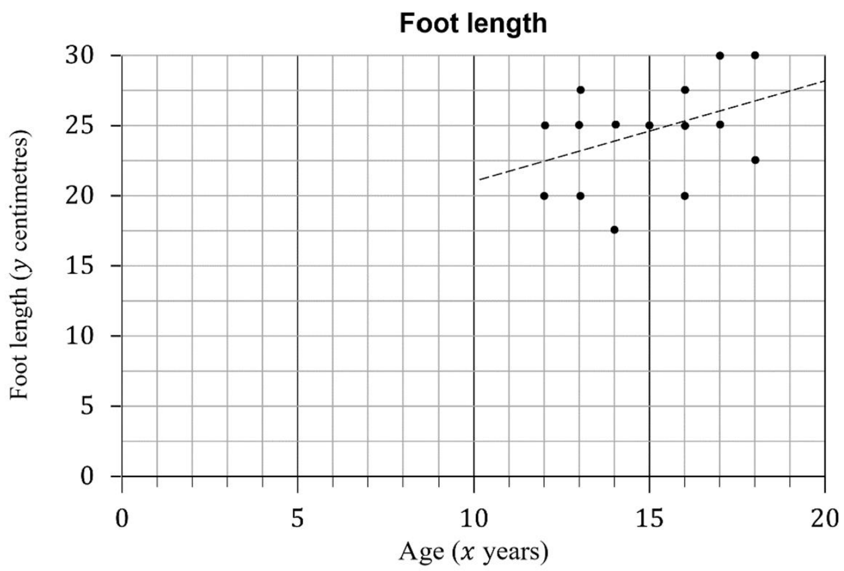
Question 25

The graph $y = \sin x$ is vertically dilated by a scale factor of 2 and horizontally dilated such that it has a period of $\frac{2\pi}{3}$.

Find the x-coordinates of the points of intersection of the transformed sine curve and the line $y = \sqrt{2}$ for $x \in [0,2\pi]$. **3**

Question 26

For a sample of fifteen high school students, their age in years, x, and the length of their right foot in centimetres, y, were recorded.



The graph shows the data as well as a regression line which passes through (12, 22.5) and (19, 27.5).

a) Find the equation of the regression line in the form $y = mx + c$. **2**

Question 26 continues on page 25

b) Layton has a five-year old brother whose right foot is 13 cm in length.

He extends the line of best fit and notes that it predicts that a five-year old should have a foot length of approximately 17.5 cm.

Give TWO reasons why Layton is incorrect to assume from the bivariate data that his brother has smaller than normal feet for a five-year old. **2**

Question 27

The random variable X has this probability distribution.

X	16	17	18	19	20
$P(X = x)$	0.2	0.4	0.1	0.2	0.1

a) Find the expected value. **1**

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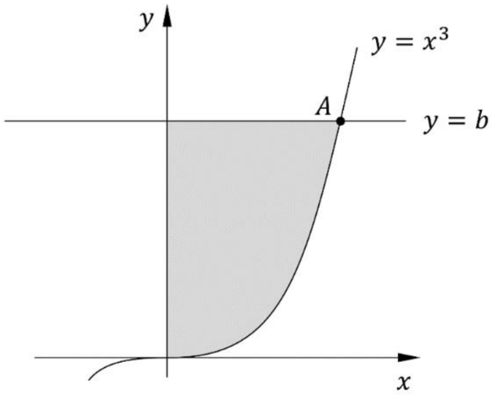
b) Find the standard deviation, correct to 2 decimal places. Show working to justify your answer. **2**

Question 28

The diagram shows the function $y = x^3$ and the line $y = b$



NOT TO SCALE

a) By solving an appropriate equation, show that the coordinates of A are $\left(\sqrt[3]{b}\,,b\right)$. **1**

----

b) By finding the area between the curve $y = x^3$, the line $y = b$ and the y-axis, or otherwise, find the value of b such that the shaded area is $\frac{4}{27}$ unit^2. **3**

APPROXIMATELY THREE QUARTERS COMPLETE – 75 marks out of 100 complete

Question 29

An astronomer is attempting to predict the height of a satellite above the ground as it orbits around the Earth. Using precise measurements, the height of the satellite can be modelled with the function:

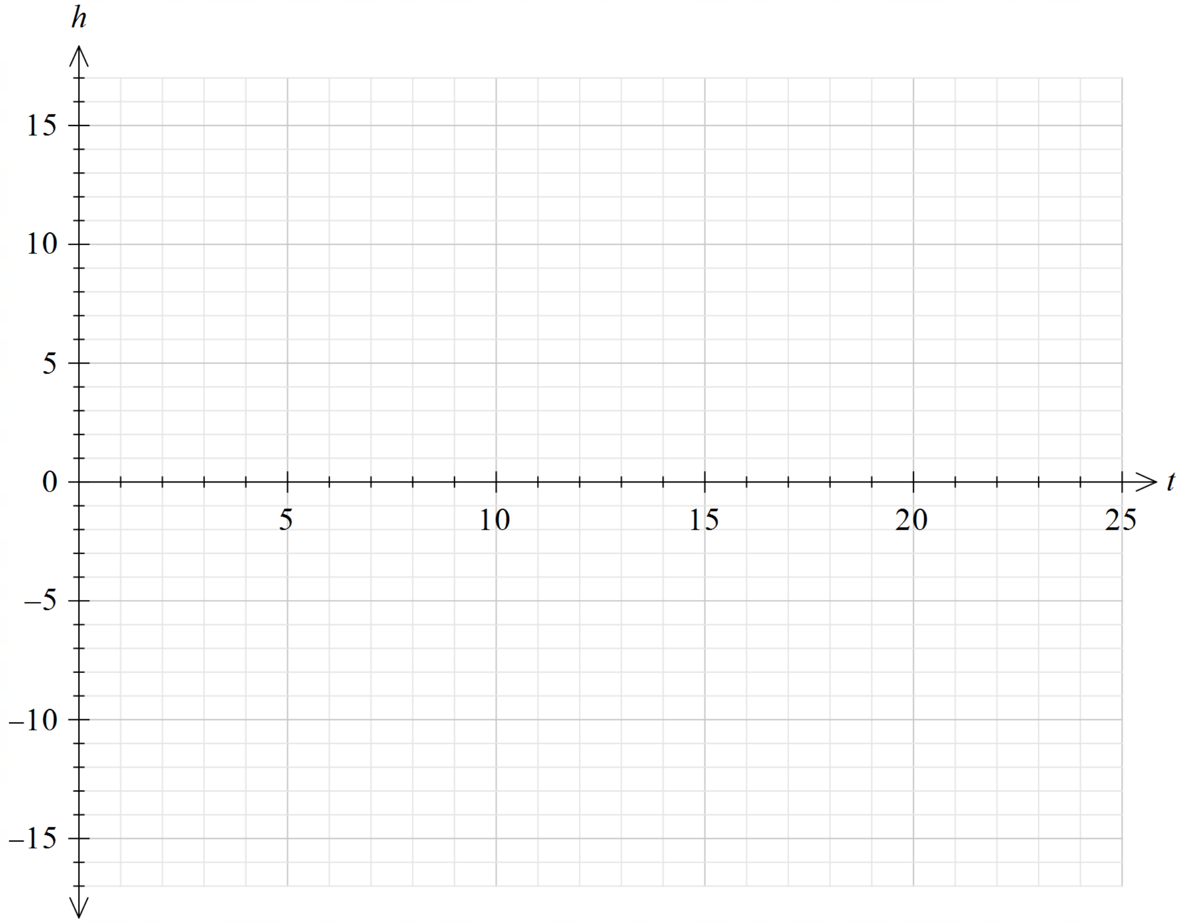
$$h(t) = 6\sin\left(\frac{\pi}{6}(t-15)\right) + 8$$

where $h(t)$ is the height of the satellite in thousands of kilometres at time t hours since midnight on any day of the year.

a) Find the minimum and maximum height reached by the satellite. **1**

----

b) On the grid below, sketch the graph of $h(t)$ for $0 \leq t \leq 24$. **2**



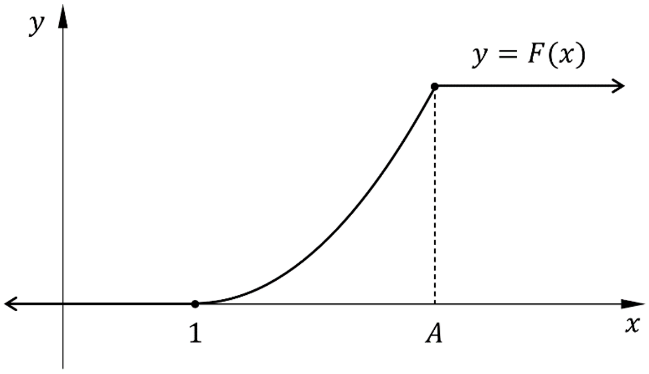
Question 28 continue on page 29

c) The satellite is out of range if it is higher than 11 000 km above the ground.
Use algebraic techniques to find how many hours in the day the satellite is out of range. **3**

End of Question 29

Question 30

The graph of a cumulative distribution function is shown below.



NOT TO SCALE

The curved section of the graph is part of the function $y = \dfrac{(x-1)^3}{6}$

a) Show that the value of A is $\sqrt[3]{6} + 1$. **2**

---

b) Find the probability density function for the given cumulative distribution function, including any restrictions on the domain. **2**

Question 31

Consider the gradient function $f'(x) = 3(x+3)(x-1)$.

a) The graph of $y = f(x)$ passes through the point $(2,-8)$. Show that $f(x) = x^3 + 3x^2 - 9x - 10$. **3**

----

b) Find the coordinates of the minimum and maximum values in the interval $-4 \leq x \leq 4$. **2**

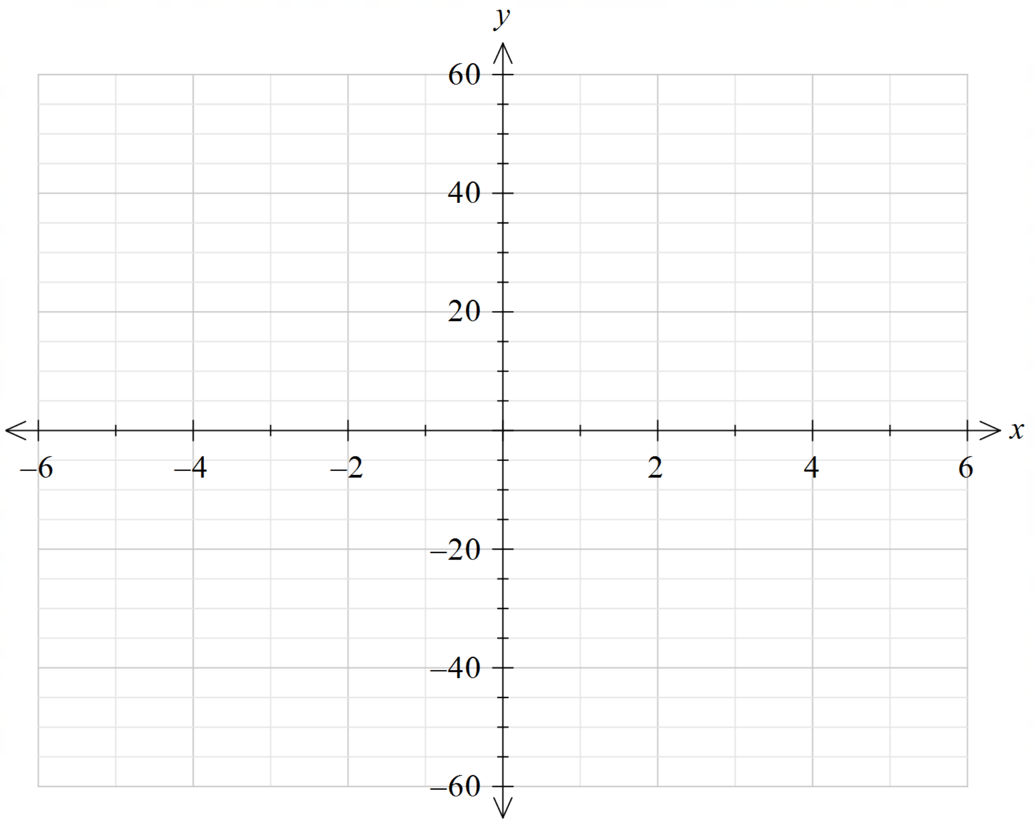
You DO NOT need to determine the nature of the stationary points.

Question 31 continues on page 32

c) Show that a point of inflection exists and find its coordinates. **2**

----

d) Sketch the graph of $y = f(x)$ in the interval $-4 \leq x \leq 4$, showing the locations of the endpoints, the stationary points and the inflection point. **2**



End of Question 31

Question 32

An electric vehicle with an empty battery is being recharged. The capacity, C%, of the battery while charging at time t minutes may be modelled by the equation:

$$C = 100(1 - 2^{-kt})$$

The battery is charged at 35% capacity after 50 minutes.

a) Show that the value of k is 0.01243, correct to 4 significant figures. **2**

----

b) To prolong the life of the battery, the charger may be set to switch off when the battery reaches 90% capacity. For how long will the battery be on charge until it reaches 90% capacity? **2**

Question 32 continue on page 34

c) By considering the first and second derivative, describe the behaviour of C for $t \geq 0$ if the battery is left on charge indefinitely. **2**

End of Examination!!! 😊

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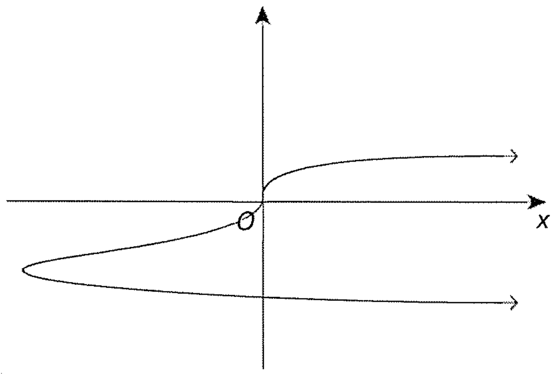


Which type of relationship does the graph represent?

(A) one-to-one

(B) one-to-many

(C) many-to-one

(D) many-to-many

Pass horizontal line test.
Fail vertical line test.
∴ one-to-many.

2 Consider the graph $y = 4x + 1$

What is the equation of the graph after it has been translated 2 units to the right?

(A) $y = 2x - 7$

(B) $y = 2x + 9$

(C) $y = 4x - 7$

(D) $y = 4x + 9$

$y = 4(x-2)+1$
$y = 4x-8+1$
$y = 4x-7$

3 Which of the following is the domain of the function $y = \dfrac{1}{\sqrt{x^2 - 1}}$?

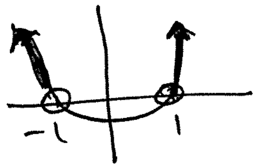
(A) $(-\infty,-1] \cup [1,\infty)$

(B) $(-\infty,-1) \cup (1,\infty)$

(C) (-1,1)

(D) [-1,1]

Domain: $x^2 - 1 > 0$



$\therefore x < -1, x > 1$

4 Families in Caringbah were surveyed about the types of pets they own. The results of the survey are shown below.

- 40% of families own a dog.
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- 18% of families own both a cat and a dog.

A family is chosen at random and found to own a cat.

What is the probability that the family also owns a dog?

(A) 18%

(B) 36%

(C) 40%

(D) 50%

$P(D) = 0.4, P(C) = 0.36$

$P(D \cap C) = 0.18$

$P(D|C) = \dfrac{P(C \cap D)}{P(C)}$

$= \dfrac{0.18}{0.36}$

$= 0.5$

$= 50\%$.

5 A supermarket assistant sets up a display of lemonade cans. The display forms a triangle, where each row has one fewer can than the one below. There are 15 cans at the base of the display and one can at the top of the display.

What is the total number of cans used in the display?

(A) 30

(B) 120

(C) 160

(D) 180

$a = 1, l = 15$

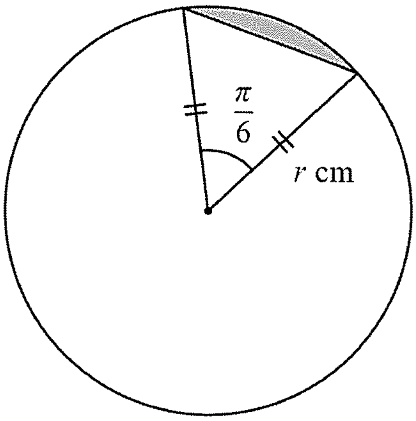
$S_n = \frac{n}{2}(1 + 15)$

$S_n = 8n$

$\therefore S_{15} = 8 \times 15$

$= 120$

6 The diagram shows a circle with radius r cm and a sector with an internal angle of $\frac{\pi}{6}$ radians.



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(A) 0.12 cm

(B) 0.34 cm

(C) 12.00 cm

(D) 144.08 cm

$A = \frac{1}{2}r^2(\theta - \sin\theta)$

$1.7 = \frac{1}{2}r^2\left(\frac{\pi}{6} - \sin\frac{\pi}{6}\right)$

$3.4 = r^2\left(\frac{\pi}{6} - \frac{1}{2}\right)$

$r^2 = \dfrac{3.4}{\frac{\pi}{6} - \frac{1}{2}}$

$r^2 = 144.075...$

$r = 12.00$ cm

7 Alicia and Steven are participating in a fitness program. The probability that Alicia will complete the program is 0.8, and the probability that Steven will complete the program is 0.7.

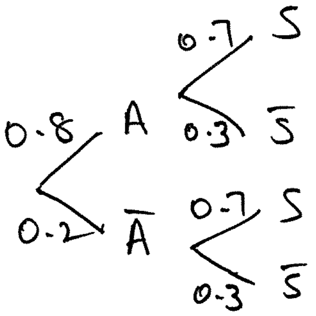
What is the probability that only one of them will successfully complete the program?

(A) 0.14

(B) 0.24

(C) 0.38

(D) 0.56



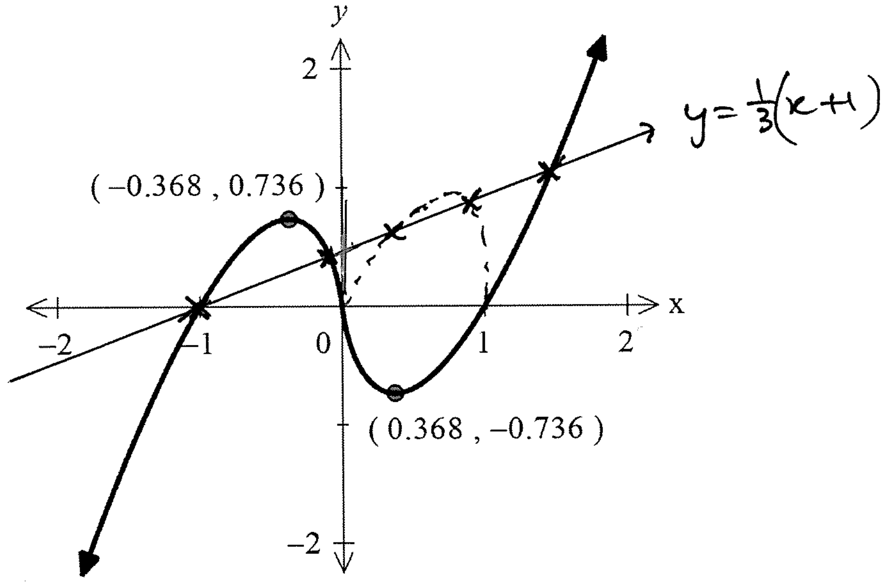
P(1 completed the program)

$= A\bar{S} + \bar{A}S$

$= (0.8 \times 0.3) + (0.2 \times 0.7)$

$= 0.38$

8 The graph of $y = x\ln(x^2)$ is shown.



How many solutions does the equation $|x\ln(x^2)| = \frac{1}{3}(x + 1)$ have?

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Height (cm)	Cumulative Frequency
131 – 140	70
141 – 150	210
151 – 160	420
161 – 170	680
171 – 180	800

What is the height of a student in the 7th decile?

(A) 131 – 140 cm

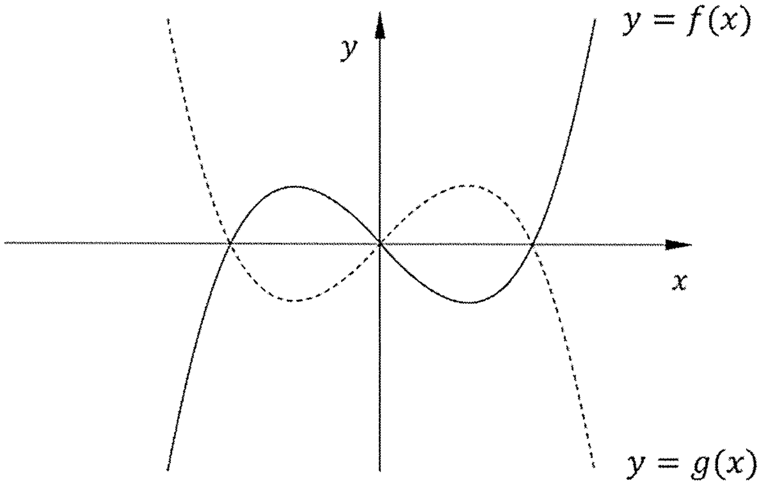
$\frac{70}{100} \times 800 = 560$

(B) 151 – 160 cm

(C) 161 – 170 cm

(D) 171 – 180 cm

10 The graph shows two cubic functions, $y = f(x)$ and $y = g(x)$.



It is given that $f(x)$ is an odd function and that $g(x) = f(-x)$.

Define $h(x) = f(x) - g(x)$.

How many stationary points does $y = h(x)$ have?

(A) 0

(B) 1

(C) 2

(D) 3

$h(x) = f(x) - g(x)$
$= f(x) - f(-x)$, since odd fnc.
$= f(x) + f(x)$
$= 2f(x)$

∴ this is a vertical dilation of $y = f(x)$, so it will have the same number of stationary points. So $y = h(x)$ has 2 stat. points.

End of Section I

Marks

Question 11

Differentiate $y = xe^{7x}$, expressing your answer in factorised form. **2**

$y' = x(7e^{7x}) + e^{7x}(1)$

$= e^{7x}(7x+1)$

Question 12

Find the exact gradient of the tangent to the curve $y = e^{2x+3}$ at the point where $x = 0$. **2**

$y' = 2e^{2x+3}$, $x=0$

when $x=0$, $y' = 2e^3$

$\therefore m_T = 2e^3$

Question 13

Solve the equation $\frac{\text{cosec}x}{5} - 2 = 0$ for $0° \le x \le 360°$, correct to the nearest minute. **3**

$\text{cosec}\,x = 10$

$\frac{1}{\sin x} = 10$

$\sin x = \frac{1}{10}$

$x = \sin^{-1}\left(\frac{1}{10}\right)$

$x = 5°44'$

$174°16'$ S | A $5°44'$
T | C

$\therefore x = 5°44'$ & $174°16'$

Question 14

A bag contains six cards that are each printed with one of the following numbers.

111, 121, 122, 211, 221, 222

A student chooses one card at random.

Let N_1 be the event that the first digit of the number on the card is 1, and N_2 be the event that the second digit of the number on the card is 1.

Determine whether N_1 and N_2 are independent events. **3**

$N_1 = \{111, 121, 122\}$ $N_2 = \{111, 211\}$

$P(N_1) = \frac{1}{2}$, $P(N_2) = \frac{1}{3}$, so $P(N_1 \cap N_2) = \frac{1}{6}$

If N_1 & N_2 are independent events, then

$P(N_1 \cap N_2) = P(N_1) \times P(N_2)$

$= \frac{1}{2} \times \frac{1}{3}$

$= \frac{1}{6}$

$= P(N_1 \cap N_2)$

$\therefore N_1$ & N_2 are independent.

Question 15

Evaluate $\int_0^{\frac{\pi}{4}} \cos(2x)\, dx$ **2**

$= \frac{1}{2}\left[\sin 2x\right]_0^{\pi/4}$

$= \frac{1}{2}\left(\sin\frac{\pi}{2} - \sin 0\right)$

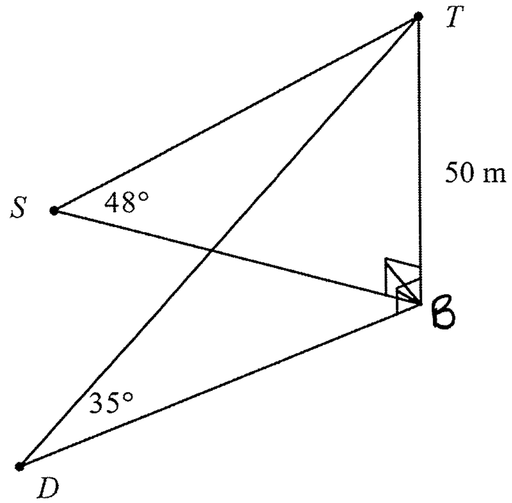
$= \frac{1}{2}(1-0)$

$= \frac{1}{2}$

Question 16

Darren and Sylvia visit Sydney Harbour on New Year's Eve to watch the fireworks.

Darren is standing outside the Sydney Opera House and Sylvia is on a yacht in the harbour. They can both see the fireworks at the top of the Sydney Harbour Bridge.



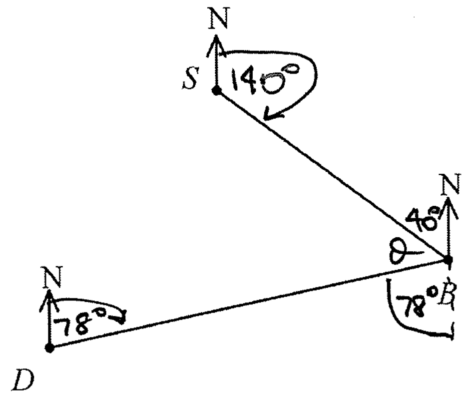
NOT TO SCALE

From Darren's position (*D*), the Sydney Harbour Bridge (*B*) is at a bearing of 78°, and the angle of elevation to the top of the bridge (*T*) is 35°.

From Sylvia's position (*S*), the Sydney Harbour Bridge is at a bearing of 140°, and the angle of elevation to the top of the bridge is 48°.

The top of the bridge is 50 m above the base of the bridge.

a) Using appropriate diagrams and reasoning, show that $\angle SBD = 62°$. **2**



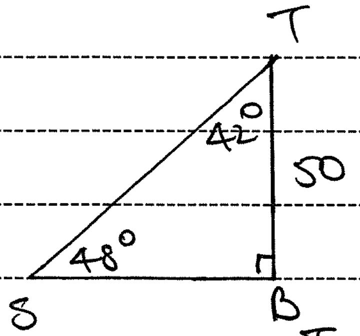
$\angle SBN = 40°$ (co-int. ∠'s =, ∥ lines)

$\angle SBD = 180° - 40° - 78°$

$= 62°$

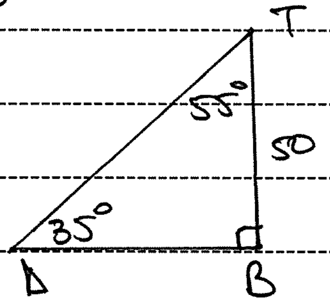
Question 16 continues on page 13

b) Hence, find the distance between Sylvia and Darren, correct to 2 decimal places. **3**



$\tan 42^\circ = \frac{SB}{50}$

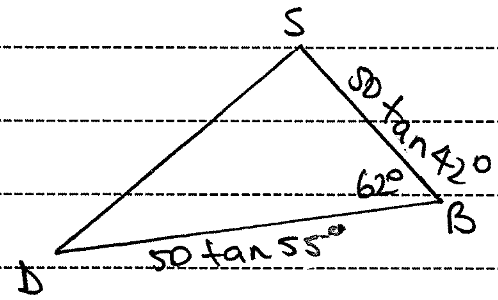
$SB = 50\tan 42^\circ$



$\tan 55^\circ = \frac{BD}{50}$

$BD = 50\tan 55^\circ$

setting both up



$SD^2 = (50\tan 55^\circ)^2 + (50\tan 42^\circ)^2 - 2(50\tan 55^\circ)(50\tan 42^\circ)\cos 62^\circ$

$= 4107.3439\ldots$

$SD = \sqrt{4107.3439\ldots}$

$\therefore SD = 64.09\text{ m}$

End of Question 16

Question 17

Find the equation of the normal to the curve $y = \dfrac{\ln(x-1)}{x-1}$ at the point $(2,0)$. **3**

$y' = \dfrac{(x-1)\left(\frac{1}{x-1}\right) - \ln(x-1)(1)}{(x-1)^2}$

$= \dfrac{1-\ln(x-1)}{(x-1)^2}$

sub $x=2$, $y' = \dfrac{1-\ln(2-1)}{(2-1)^2}$

$= 1 - \ln(1)$

$= 1$

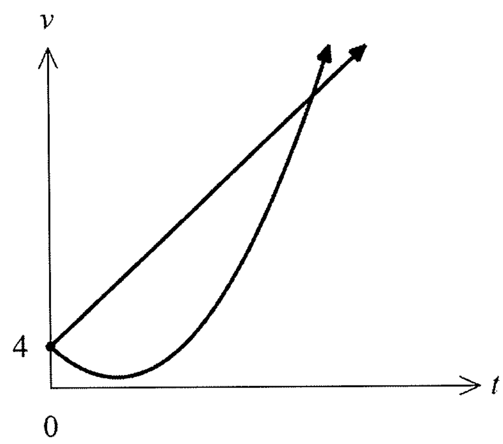
$\therefore m_T = 1$ & $m_N = -1$

Eqn of normal: $y - 0 = -1(x-2)$

$y = -x + 2$

Question 18

The velocity of two racing cars, P and Q, are shown on the graph below.



NOT TO SCALE

Question 18 continues on page 15

The velocity of car P at time t seconds is given by the function $v_P = 3t^2 - 6t + 4$, where v_P is the velocity in metres per second.

The velocity of car Q is accelerating at a constant rate. Both cars start at the same point and have the same velocity at times where $t = 0$ and $t = 4$.

a) Show that the equation for the velocity of car Q, v_Q, is given by $v_Q = 6t + 4$. **2**

At $t=0$, $V_Q = V_P = 4$

At $t=4$, $V_Q = V_P = 3(4)^2 - 6(4) + 4 = 28$

$\therefore (0,4)$ & $(4,28)$

$m = \frac{28-4}{4-0} = 6$

$\therefore V_Q = 6t+4$

b) Both cars start the race from the same point. **3**

Find the earliest time when car P will pass car Q after the race starts.

$\int_0^a 3t^2 - 6t + 4 - (6t+4)\, dt = 0$

$\int_0^a (3t^2 - 12t)\, dt = 0$

$\left[t^3 - 6t^2\right]_0^a = 0$

$a^3 - 6a^2 = 0$

$a^2(a-6) = 0$

$a = 0, 6$

$\therefore$ car P will pass Q after 6 seconds from the start of the race.

End of Question 18

Question 19

a) Find the derivative of $y = \tan(x^2)$. **1**

$y' = 2x\sec^2(x^2)$

b) Hence, or otherwise, find $\int x\tan^2(x^2)\,dx$. **3**

$\int x\tan^2(x^2)\,dx = \int x(\sec^2(x^2) - 1)\,dx$.

$= \frac{1}{2}\int 2(x\sec^2(x^2) - 1)\,dx$

$= \frac{1}{2}\int 2x\sec^2(x^2) - 2x\,dx$.

$= \frac{1}{2}(\tan(x^2) - x^2) + C$

Question 20

Use two applications of the trapezoidal rule to find an approximate value of $\int_1^2 \ln x\,dx$. **2**

Give your answer correct to 2 decimal places.

$\int_1^2 \ln x\,dx \doteq \frac{0.25}{2}\left[(0 + \ln 2) + 2(\ln(1.25) + \ln(1.5) + \ln(1.75))\right]$

$= 0.383699\ldots$

$= 0.38$

Question 21

a) Show that $\dfrac{1}{\sec\theta - \tan\theta} - \dfrac{1}{\sec\theta + \tan\theta} = 2\tan\theta$ **2**

LHS: $\dfrac{1}{\sec\theta - \tan\theta} - \dfrac{1}{\sec\theta + \tan\theta}$

$= \dfrac{\sec\theta + \tan\theta - (\sec\theta - \tan\theta)}{\sec^2\theta - \tan^2\theta}$

$= \dfrac{\sec\theta + \tan\theta - \sec\theta + \tan\theta}{1}$

$= 2\tan\theta$

$=$ RHS

b) Hence or otherwise, find $\displaystyle\int \frac{1}{\sec\theta - \tan\theta} - \frac{1}{\sec\theta + \tan\theta}\, d\theta$ **2**

$\displaystyle\int 2\tan\theta \, d\theta = 2\int -\frac{-\sin\theta}{\cos\theta}\, d\theta$

$= -2\ln|\cos\theta| + c$

Question 22

A continuous random variable has the probability density function $f(x)$ given by:

$$f(x) = \begin{cases} \dfrac{kx}{5-x^2} & 0 \le x \le 2 \\ 0 & \text{for all other values of } x \end{cases}$$

a) Show that the value of k is $\dfrac{2}{\ln 5}$. **2**

$$\int_0^2 f(x)\,dx = 1$$

$$\int_0^2 \frac{kx}{5-x^2}\,dx = 1$$

$$-\frac{k}{2}\int_0^2 \frac{-2x}{5-x^2}\,dx = 1$$

$$-\frac{k}{2}\left[\ln|5-x^2|\right]_0^2 = 1$$

$$-\frac{k}{2}\left[\ln|5-4| - \ln|5-0|\right] = 1$$

$$-\frac{k}{2}\left[\ln(1) - \ln(5)\right] = 1$$

$$-\frac{k}{2}(0-\ln 5) = 1$$

$$\frac{k\ln 5}{2} = 1$$

$$k = \frac{2}{\ln 5}$$

Question 22 continues on page 19

b) Show that the cumulative distribution function $F(x)$ is given by: **2**

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - \log_5(5 - x^2) & 0 \le x \le 2 \\ 1 & x > 2 \end{cases}$$

$$\int_0^x \frac{2}{\ln 5}\left(\frac{x}{5-x^2}\right) dx$$

$$= \frac{-1}{\ln 5}\int_0^x \frac{-2x}{5-x^2}\,dx$$

$$= \frac{-1}{\ln 5}\Big[\ln|5-x^2|\Big]_0^x$$

$$= \frac{1}{\ln 5}\Big[\ln|5-x^2|\Big]_x^0$$

$$= \frac{1}{\ln 5}\Big[\ln|5| - \ln|5-x^2|\Big]$$

$$= 1 - \frac{\ln|5-x^2|}{\ln 5}$$

$$= 1 - \frac{\dfrac{\log_5|5-x^2|}{\log_5 e}}{\dfrac{\log_5 5}{\log_5 e}}$$

$$= 1 - \log_5|5-x^2|$$

$$= 1 - \log_5(5-x^2), \text{ since } 5-x^2 > 0.$$

Question 22 continues on page 20

c) Hence, or otherwise, show that the median is $\sqrt{5-\sqrt{5}}$. **2**

$1-\log_5(5-x^2)=0.5$

$\log_5(5-x^2)=0.5$

$5^{0.5}=5-x^2$

$5-x^2=\sqrt{5}$

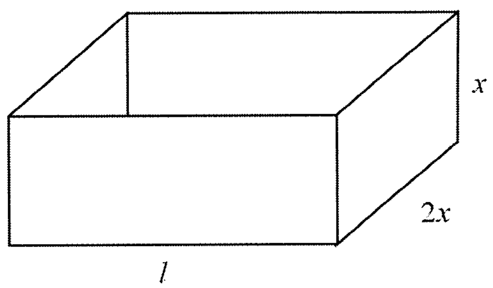
$x^2=5-\sqrt{5}$

$x=\pm\sqrt{5-\sqrt{5}}$

$\therefore x=\sqrt{5-\sqrt{5}}$, since $0\leq x\leq 2$.

Question 23

A factory produces boxes in the shape of an open rectangular prism (without a lid).



NOT TO SCALE

The length, height and width of the box are l cm, x cm and $2x$ cm respectively.

The factory uses k cm^2 of sheet metal to make the box, where k is a constant.

a) Show that $l=\dfrac{k-4x^2}{4x}$. **1**

$k=2lx+2lx+2(2x^2)$

$k=4lx+4x^2$

$4lx=k-4x^2$

$l=\dfrac{k-4x^2}{4x}$

Question 23 continues on page 21

b) Hence, find the maximum volume of the box that can be produced from a sheet of metal with an area of 1200 cm^2. **3**

$V = 2x^2 l$

$= 2x^2\left(\frac{1200 - 4x^2}{4x}\right)$

$= 600x - 2x^3$

$V' = 600 - 6x^2$

$600 - 6x^2 = 0$

$6x^2 = 600$

$x^2 = 100$

$x = \pm 10$

$V'' = -12x$

when $x = 10$, $V'' = -120$

< 0, max

when $x = -10$, $V'' = 120$

> 0, min.

$\therefore$ max volume $= 2 \times 10^2 \times \frac{1200 - 4(10)^2}{40}$

$= 4000 \text{ cm}^3$

End of Question 23

APPROXIMATELY HALFWAY – 55 marks out of 100 complete at this point.

Question 24

Consider the series $\ln x - 3(\ln x)^2 + 9(\ln x)^3 - 27(\ln x)^4 + \ldots$

a) Show that this series is a geometric series. **2**

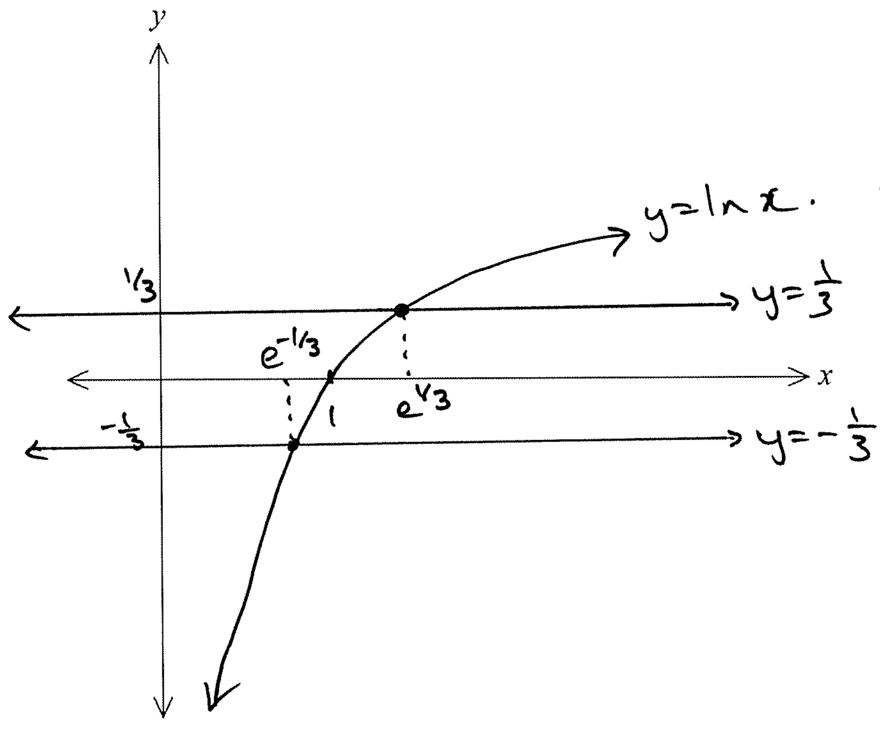
$\frac{T_2}{T_1} = \frac{-3(\ln x)^2}{\ln x}$, $\frac{T_3}{T_2} = \frac{9(\ln x)^3}{-3(\ln x)^2}$

$= -3(\ln x)$ $\quad = -3(\ln x)$

$\therefore r = -3(\ln x)$

$\therefore$ series is geometric.

b) On the set of axes below, sketch the graphs of $y = \ln x$, $y = -\frac{1}{3}$ and $y = \frac{1}{3}$, showing all points of intersection and intercepts with the coordinate axes. **2**



shape detail

Question 24 continues on page 23

c) Hence, or otherwise, find the values of x for which the series has a limiting sum. **2**

$|-3(\ln x)| < 1$

$-3(\ln x) < 1$ or $3(\ln x) < 1$

$\ln x > \frac{-1}{3}$ $\quad$ $\ln x < \frac{1}{3}$

$\log_e x > \frac{-1}{3}$ $\quad$ $\log_e x < \frac{1}{3}$

$x > e^{1/3}$ $\quad$ $x < e^{1/3}$

$\therefore e^{-1/3} < x < e^{1/3}$

Question 25

The graph $y = \sin x$ is vertically dilated by a scale factor of 2 and horizontally dilated such that it has a period of $\frac{2\pi}{3}$.

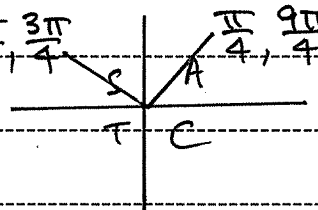
Find the x-coordinates of the points of intersection of the transformed sine curve and the line $y = \sqrt{2}$ for $x \in [0, 2\pi]$. **3**

Period $= \frac{2\pi}{n} = \frac{2\pi}{3}$, $\therefore n = 3$ $\quad [0, 2\pi]$

A vertical dilation by a factor of 2 gives an amplitude of 2, so $y = 2\sin 3x$ for $0 \leq 3x \leq 6\pi$

sub $y = \sqrt{2}$, $2\sin 3x = \sqrt{2}$ for $0 \leq 3x \leq 6\pi$

$\sin 3x = \frac{\sqrt{2}}{2}$ $\quad$ $\frac{19\pi}{4}, \frac{11\pi}{4}, \frac{3\pi}{4}$ $\quad$ $\frac{\pi}{4}, \frac{9\pi}{4}, \frac{17\pi}{4}$

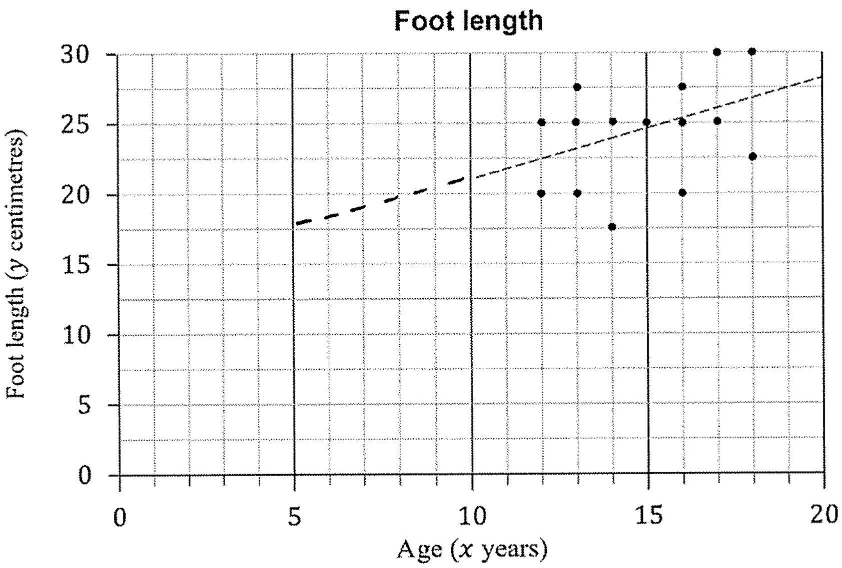


$\therefore 3x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, \frac{17\pi}{4}, \frac{19\pi}{4}$

$x = \frac{\pi}{12}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{11\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}$

Question 26

For a sample of fifteen high school students, their age in years, x, and the length of their right foot in centimetres, y, were recorded.



The graph shows the data as well as a regression line which passes through (12, 22.5) and (19, 27.5).

a) Find the equation of the regression line in the form $y = mx + c$. **2**

$x_1 \ y_1 \quad x_2 \ y_2$

$(12, 22.5) \quad (19, 27.5)$

$m = \frac{27.5 - 22.5}{19 - 12}$

$= \frac{5}{7}$

$y - 22.5 = \frac{5}{7}(x - 12)$

$y - 22.5 = \frac{5}{7}x - \frac{60}{7}$

$y = \frac{5}{7}x + \frac{195}{14}$

Question 26 continues on page 25

b) Layton has a five-year old brother whose right foot is 13 cm in length.

He extends the line of best fit and notes that it predicts that a five-year old should have a foot length of approximately 17.5 cm.

Give TWO reasons why Layton is incorrect to assume from the bivariate data that his brother has smaller than normal feet for a five-year old. **2**

- Layton is extrapolating well outside the dataset as the students measured were aged from 12 to 18, which is always problematic as it assumes a linear relationship which may not be true.
- The length of the relationship between age and foot length is low (the correlation coefficient is closed to 0), so even interpolating within the correct age range would be incorrect.

Question 27

The random variable X has this probability distribution.

X	16	17	18	19	20
$P(X = x)$	0.2	0.4	0.1	0.2	0.1

a) Find the expected value. **1**

$E(x) = 16(0.2) + 17(0.4) + 18(0.1) + 19(0.2) + 20(0.1)$

$\therefore \mu = 17.6$

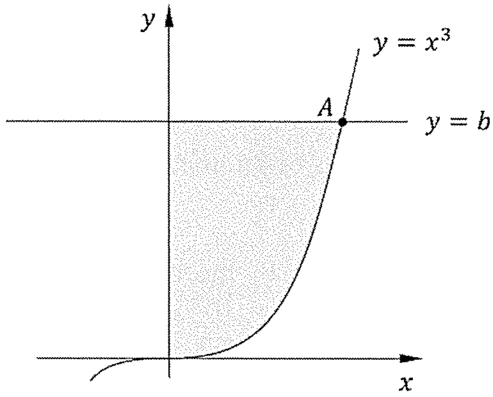
b) Find the standard deviation, correct to 2 decimal places. Show working to justify your answer. **2**

$Var(x) = (16-17.6)^2(0.2) + (17-17.6)^2(0.4) + (18-17.6)^2(0.1)$
$+ (19-17.6)^2(0.2) + (20-17.6)^2(0.1)$
$= 1.64$ OR $\frac{41}{25}$.

$\sigma = \sqrt{1.64}$
$= 1.28$ (2dp) OR $\frac{\sqrt{41}}{5}$

Question 28

The diagram shows the function $y = x^3$ and the line $y = b$



NOT TO SCALE

a) By solving an appropriate equation, show that the coordinates of A are $\left(\sqrt[3]{b}, b\right)$. **1**

$y = b, \quad y = x^3$

$x^3 = b$

$x = \sqrt[3]{b}$

$\therefore A\left(\sqrt[3]{b}, b\right)$

b) By finding the area between the curve $y = x^3$, the line $y = b$ and the y-axis, or otherwise, find the value of b such that the shaded area is $\frac{4}{27}$ unit^2. **3**

$y = x^3$

$x = y^{1/3}$

$A = \int_0^b y^{1/3}\, dy$

$\frac{4}{27} = \left[\frac{3y^{4/3}}{4}\right]_0^b$

$\frac{3b^{4/3}}{4} = \frac{4}{27}$

$3b^{4/3} = \frac{16}{27}$

$b^{4/3} = \frac{16}{81}$

$b = \frac{8}{27}$

APPROXIMATELY THREE QUARTERS COMPLETE – 75 marks out of 100 complete

Question 29

An astronomer is attempting to predict the height of a satellite above the ground as it orbits around the Earth. Using precise measurements, the height of the satellite can be modelled with the function:

$$h(t) = 6\sin\left(\frac{\pi}{6}(t-15)\right) + 8$$

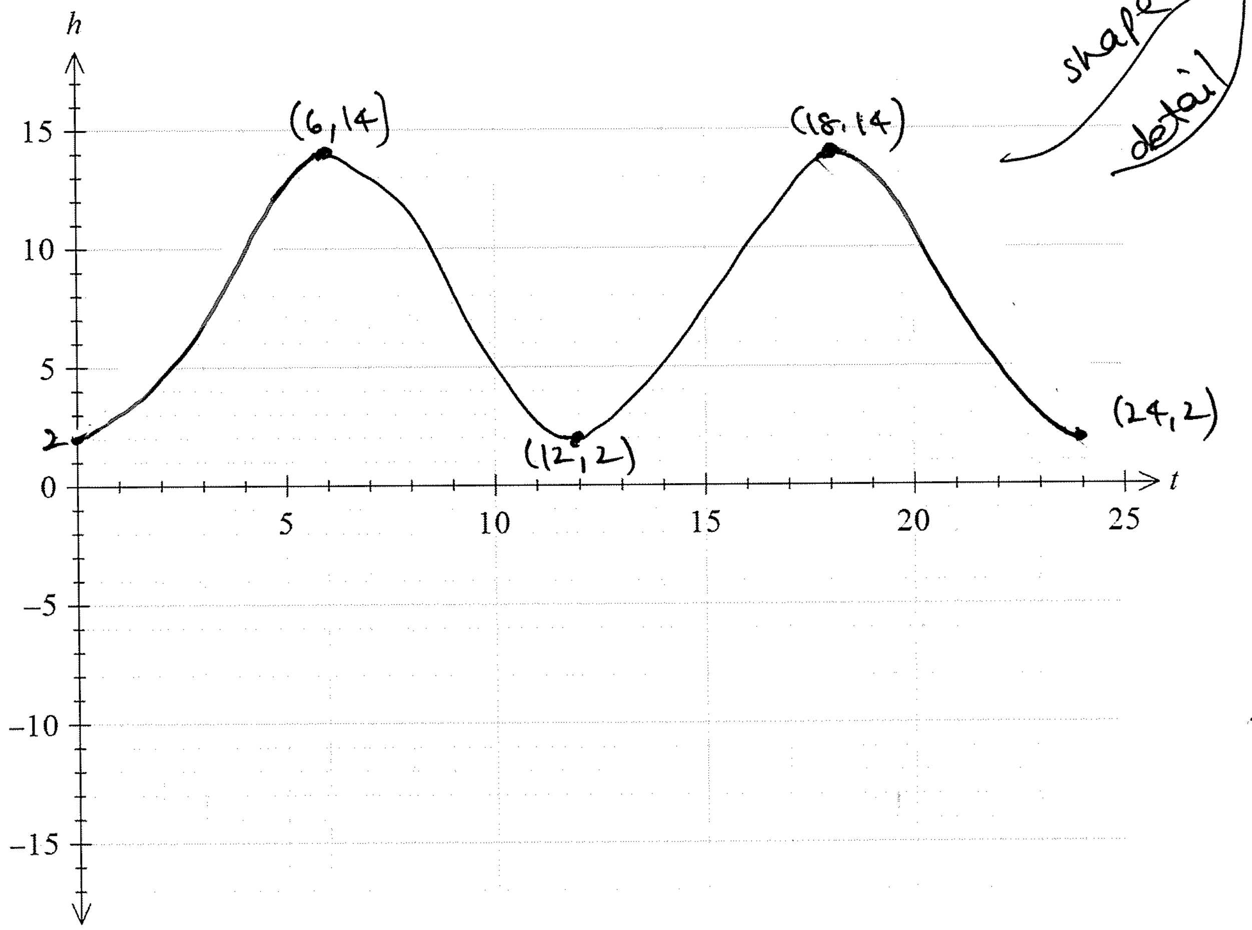
where $h(t)$ is the height of the satellite in thousands of kilometres at time t hours since midnight on any day of the year.

a) Find the minimum and maximum height reached by the satellite. **1**

min = (8−6) × 1000 = 2000 km

max = (8+6) × 1000 = 14000 km

b) On the grid below, sketch the graph of $h(t)$ for $0 \leq t \leq 24$. **2**



Question 28 continue on page 29

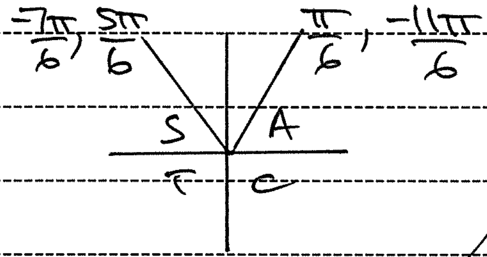
c) The satellite is out of range if it is higher than 11 000 km above the ground.
Use algebraic techniques to find how many hours in the day the satellite is out of range. **3**

$h(t) = 11$

$6\sin\left(\frac{\pi}{6}(t-15)\right)+8 = 11$

$6\sin\left(\frac{\pi}{6}(t-15)\right) = 3$

$\sin\left(\frac{\pi}{6}(t-15)\right) = \frac{1}{2}$



$\frac{\pi}{6}(t-15) = \frac{\pi}{6}$

$\therefore \frac{\pi}{6}(t-15) = \frac{\pi}{6}, \frac{5\pi}{6}, -\frac{7\pi}{6}, -\frac{11\pi}{6}$

$t-15 = 1, 5, -7, -11$

$t-15 = -11, -7, 1, 5$

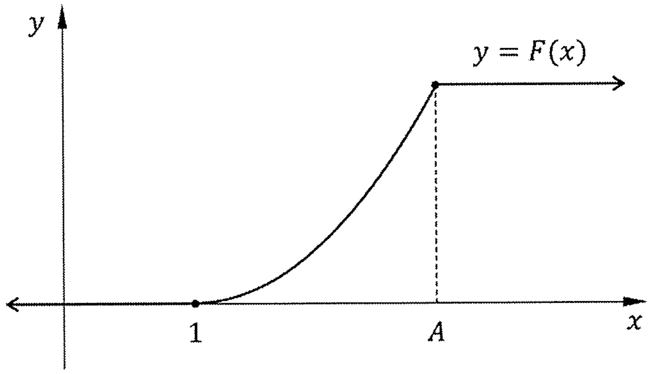
$\therefore t = 4, 8, 16, 20.$

So, from the graph, the satellite is above 11000km in the interval $4<t<8$ & $16<t<20$.
$\therefore$ satellite is out of range for 8 hours each day.

End of Question 29

Question 30

The graph of a cumulative distribution function is shown below.



NOT TO SCALE

The curved section of the graph is part of the function $y = \dfrac{(x-1)^3}{6}$

a) Show that the value of A is $\sqrt[3]{6} + 1$. **2**

Let $x = A$, $1 = \dfrac{(A-1)^3}{6}$

$(A-1)^3 = 6$

$A - 1 = \sqrt[3]{6}$

$A = \sqrt[3]{6} + 1$

b) Find the probability density function for the given cumulative distribution function, including any restrictions on the domain. **2**

For $1 \leq x \leq \sqrt[3]{6} + 1$

$f(x) = \dfrac{d}{dx}\left(F(x)\right)$

$= \dfrac{d}{dx}\left(\dfrac{x-1}{6}\right)^3$

$= \dfrac{(x-1)^2}{2}.$

$\therefore f(x) = \begin{cases} \dfrac{(x-1)^2}{2}, & 1 \leq x \leq \sqrt[3]{6} + 1 \\ 0 & \end{cases}$

c) Show that a point of inflection exists and find its coordinates. **2**

$f'(x) = 3x^2 + 6x - 9$

$f''(x) = 6x + 6$

For inflection point:

$6x + 6 = 0$

$6x = -6$

$x = -1$

$\therefore (-1, 1)$ is a possible inflection pt.

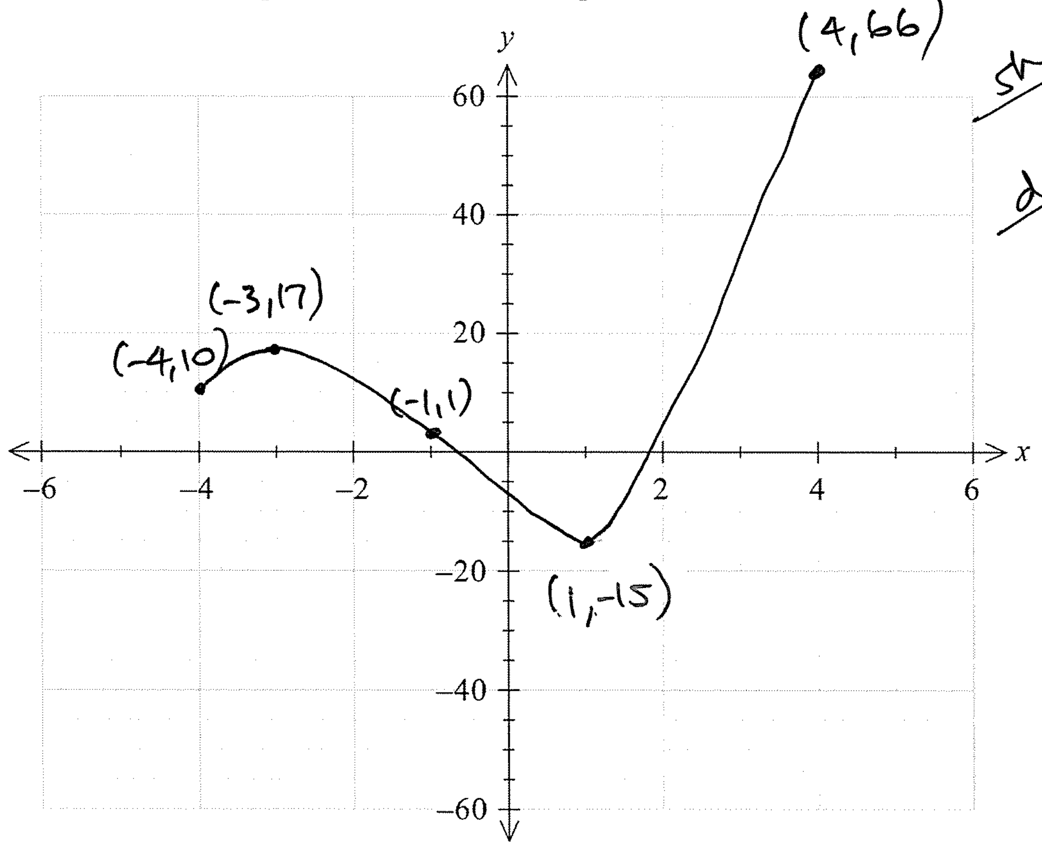
check concavity

x	-2	-1	0
$f''(x)$	-6	0	6

$\therefore$ change in concavity

$\therefore (-1,1)$ is an inflection pt.

d) Sketch the graph of $y = f(x)$ in the interval $-4 \leq x \leq 4$, showing the locations of the endpoints, the stationary points and the inflection point. **2**



shape

detail

End of Question 31

Question 31

Consider the gradient function $f'(x) = 3(x + 3)(x - 1)$.

a) The graph of $y = f(x)$ passes through the point $(2,-8)$. Show that $f(x) = x^3 + 3x^2 - 9x - 10$. **3**

$f(x) = 3\int(x^2+2x-3)\,dx$

$= 3\left[\frac{x^3}{3}+x^2-3x\right]+c$

$\therefore f(x) = x^3+3x^2-9x+c$

sub $(2,-8)$

$-8 = 2^3+3(2)^2-9(2)+c$

$-8 = 2+c$

$c = -10$

$\therefore f(x) = x^3+3x^2-9x-10$

b) Find the coordinates of the minimum and maximum values in the interval $-4 \leq x \leq 4$. **2**

You DO NOT need to determine the nature of the stationary points.

stationary points are at $x=-3$ and $x=1$

$f(-4)=10$, $f(-3)=17$, $f(1)=-15$, $f(4)=66$.

$\therefore$ min value is $(1,-15)$ & max value is $(4,66)$

Question 31 continues on page 32

Question 32

An electric vehicle with an empty battery is being recharged. The capacity, $C\%$, of the battery while charging at time t minutes may be modelled by the equation:

$$C = 100(1 - 2^{-kt})$$

The battery is charged at 35% capacity after 50 minutes.

a) Show that the value of k is 0.01243, correct to 4 significant figures. **2**

$35 = 100(1 - 2^{-50t})$

$0.35 = 1 - 2^{-50t}$

$2^{-50t} = 0.65$

$\ln(2)^{-50t} = \ln(0.65)$

$-50t\ln(2) = \ln(0.65)$

$t = \dfrac{\ln(0.65)}{-50\ln(2)}$

$t = 0.01243$

b) To prolong the life of the battery, the charger may be set to switch off when the battery reaches 90% capacity. For how long will the battery be on charge until it reaches 90% capacity? **2**

$90 = 100(1 - 2^{-0.01243t})$

$0.9 = 1 - 2^{-0.01243t}$

$2^{-0.01243t} = 0.1$

$\ln(2)^{-0.01243t} = \ln(0.1)$

$-0.01243t\ln(2) = \ln(0.1)$

$t = \dfrac{\ln(0.1)}{-0.01243(\ln 2)}$

$t = 267.2508\ldots$ mins

$t \doteqdot 4$h 27mins

Question 32 continue on page 34

c) By considering the first and second derivative, describe the behaviour of C for $t \geq 0$ if the battery is left on charge indefinitely. **2**

Initially, $c=0 \Rightarrow 2^{-kt} \to 0$ as $t \to \infty$ for $k>0$.

$\therefore C \to 100$ as $t \to \infty$.

$C' = 100k\ln(2) \times 2^{-kt}$ & $C'' = -100k^2(\ln 2)^2 \times 2^{-kt}$

Since $k>0$, $C'>0$ & $C''<0$ for all $t>0$.

Hence, the battery charge capacity increases from 0 at a decreasing rate over time, but can never reach 100% charge.

End of Examination!!! ☺

STUDENT NAME/NUMBER: Answers.

Section I

10 Marks

Attempt Questions 1-10.

Allow about 15 minutes for this section.

Select either A, B, C or D that best answers the question and indicate your choice with a cross (X) in the appropriate space on the grid below.

This page **must** be handed in with your answer booklet.

	A	B	C	D
1		X		
2			X	
3		X		
4				X
5		X		
6			X	
7			X	
8			X	
9			X	
10			X	